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# COMPACT KAEHLER MANIFOLDS WITH CONSTANT GENERALIZED SCALAR CURVATURE

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## 1. Introduction

Let $M$ be an $n$-dimensional Kaehler manifold with fundamental 2-form $\Phi$ and Ricci 2-form $\gamma$. Following [2], we call $M$ a *cohomological Einstein manifold* if $[\gamma] = a \cdot [\Phi]$ for some $a$, where $[*]$ denotes the cohomology class represented by $*$. The first Chern class of $M$ is represented by $\gamma$.

It is well-known that *a compact cohomological Einstein Kaehler manifold is Einsteinian if the scalar curvature is constant.* The purpose of this paper is to generalize this result.

In [2], the second author introduced the notion of *generalized scalar curvatures*: Let $\omega^1, \cdots, \omega^n$ be a local field of unitary coframes, so that the Kaehler metric of $M$ is given by $g = \frac{1}{2} \sum (\omega^\alpha \otimes \bar{\omega}^\alpha + \bar{\omega}^\alpha \otimes \omega^\alpha)$. Let $S = \frac{1}{2} \sum (R_{\alpha\beta} \omega^\alpha \otimes \bar{\omega}^\beta + \bar{R}_{\alpha\beta} \bar{\omega}^\alpha \otimes \omega^\beta)$ be the Ricci tensor of $M$. We define $n$ scalars $\rho_1, \cdots, \rho_n$ by

$$\det (\delta_{\alpha\beta} + tR_{\alpha\bar{\beta}}) = 1 + \sum_{k=1}^{n} \rho_k t^k .$$

If we denote the scalar curvature of $M$ by $\rho$, then it is easily seen that $\rho = 2\rho_1$. It is also clear that $\rho_n = \det (R_{\alpha\bar{\beta}})$.

Our main theorem is the following.

**Theorem.** *Let $M$ be an $n$-dimensional compact Kaehler manifold* $(n \geq 2)$. *If*

(i) $\rho_k$ *is constant,*

(ii) $[\gamma^k] = a \cdot [\Phi^k]$ *for some* $a$,

(iii) rank $(R_{\alpha\bar{\beta}}) \geq k$ (*or equivalently* $\rho_k \neq 0$) *for some* $k < n$,

*then $M$ is Einsteinian.*

Assumption (iii) is redundant if $k = 1$, but it is essential if $k > 1$. Immediately from the above theorem we have

**Corollary.** *Let $M$ be an $n$-dimensional compact Kaehler manifold* $(n \geq 2)$. *If*

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(i) $\rho_k$ *is constant,*
(ii) *M is cohomologically Einsteinian,*
(iii) rank $(R_{\alpha\bar\beta}) \geq k$ *(or equivalently* $\rho_k \neq 0$*) for some* $k < n$,
*then M is Einsteinian.*

## 2. Proof of the theorem

Let $\Phi$ be the fndamental 2-form of $M$, that is, a closed 2-form defined by

$$\Phi = \tfrac{1}{2}\sqrt{-1} \sum \omega^\alpha \wedge \bar\omega^\alpha .$$

Let $\gamma$ be the Ricci 2-form of $M$, that is, a closed 2-form defined by

$$\gamma = \frac{\sqrt{-1}}{4\pi} \sum R_{\alpha\bar\beta}\omega^\alpha \wedge \bar\omega^\beta .$$

Let $\Lambda$ be the operator of interior product by $\Phi$. Then we have (cf. [2])

$$\Lambda^k\gamma^k = \frac{k!k!}{(2\pi)^k}\rho_k ,$$

which, together with assumption (i), implies that

$$d\Lambda^k\gamma^k = 0 . \tag{1}$$

Let $\delta$ be the codifferential operator, and $C$ the operator defined by $C\alpha = (\sqrt{-1})^{r-s}\alpha$, where $\alpha$ is a form of bidegree $(r, s)$. Then they satisfy $d\Lambda^k - \Lambda^k d = kC^{-1}\delta C\Lambda^{k-1}$. Therefore from (1) and the fact that $\gamma$ is closed we obtain

$$\delta\Lambda^{k-1}\gamma^k = 0 . \tag{2}$$

We prove the following general lemma.

**Lemma.** *Let* $\eta$ *be a form of bidegree* $(p, q)$ *with* $p > 1$ *and* $q > 1$. *If* $\Lambda\delta\eta = 0$, *then* $\delta\eta = 0$.

*Proof.* If we denote the star isomorphism by $*$, then $\Lambda\delta\eta = 0$ is equivalent to $*\Lambda*d*\eta = 0$. If we denote the dual operator of $\Lambda$ by $L$, then $*\Lambda*d*\eta = 0$ is equivalent to $Ld*\eta = 0$. Since $L$ is an isomorphism (cf. for example [1]), $Ld*\eta = 0$ is equivalent to $d*\eta = 0$. Applying $*$ we obtain $*d*\eta = 0$ which is equivalent to $\delta\eta = 0$. q.e.d.

Since $\Lambda\delta = \delta\Lambda$, it follows from (2) that $\Lambda\delta\Lambda^{k-2}\gamma^k = 0$. Therefore by Lemma we have

$$\delta\Lambda^{k-2}\gamma^k = 0 .$$

Repeatedly applying this process we finally obtain

$$\delta\gamma^k = 0 ,$$

which, together with the fact that $d\gamma^k = 0$, implies that $\gamma^k$ is harmonic. Therefore assumption (ii) yields that

$$\gamma^k = a\Phi^k . \tag{3}$$

At each point of $M$, we can choose a unitary coframe $\omega^1, \cdots, \omega^n$ with respect to which $(R_{\alpha\beta})$ is of the form

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

so that $\gamma = \dfrac{\sqrt{-1}}{4\pi} \sum \lambda_\alpha \omega^\alpha \wedge \bar{\omega}^\alpha$. Therefore

$$\gamma^k = \left(\frac{\sqrt{-1}}{4\pi}\right)^k k! \sum_{\alpha_1 < \cdots < \alpha_k} \lambda_{\alpha_1} \cdots \lambda_{\alpha_k} \omega^{\alpha_1} \wedge \bar{\omega}^{\alpha_1} \wedge \cdots \wedge \omega^{\alpha_k} \wedge \bar{\omega}^{\alpha_k} .$$

Hence from (3) we obtain the following system of simultaneous equations

$$\begin{aligned} &\lambda_1 \cdots \lambda_{k-1}(\lambda_k - \lambda_{k+1}) = 0 , \\ &\lambda_1 \cdots \lambda_{k-1}(\lambda_k - \lambda_{k+2}) = 0 , \\ &\cdots\cdots\cdots\cdots , \end{aligned} \tag{4}$$

which consists of $\binom{n}{k-1}\binom{n-k+1}{2}$ equations. It is easily seen that, under assumption (iii), (4) implies that $\lambda_1 = \cdots = \lambda_n$. Therefore $M$ is Einsteinian.

**Remark.** Assumption (iii) is essential if $k>1$. In fact, let $M = P_{k-1}(C) \times T^{n-k+1}$, where $P_{k-1}(C)$ denotes a $(k-1)$-dimensional complex projective space with the Fubini-Study metric, and $T^{n-k+1}$ denotes an $(n-k+1)$-dimensional complex torus with the flat metric. Then $M$ satisfies assumptions (i) and (ii), but $M$ is not Einsteinian.

## References

[1] S. I. Goldberg, *Curvature and homology*, Academic Press, New York, 1962.
[2] K. Ogiue, *Generalized scalar curvatures of cohomological Einstein Kaehler manifolds*, J. Differential Geometry **10** (1975) 201–205.

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